Rayleigh and Prandtl number scaling in the bulk of Rayleigh–Bénard turbulence

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(Received 9 October 2004; accepted 1 February 2005; published online 5 May 2005)

The Ra and Pr number scaling of the Nusselt number Nu, the Reynolds number Re, the temperature fluctuations, and the kinetic and thermal dissipation rates is studied for (numerical) homogeneous Rayleigh–Bénard turbulence, i.e., Rayleigh–Bénard turbulence with periodic boundary conditions in all directions and a volume forcing of the temperature field by a mean gradient. This system serves as model system for the bulk of Rayleigh–Bénard flow and therefore as model for the so-called “ultimate regime of thermal convection.” With respect to the Ra dependence of Nu and Re we confirm our earlier results [D. Lohse and F. Toschi, “The ultimate state of thermal convection,” Phys. Rev. Lett. 90, 034502 (2003)] which are consistent with the Kraichnan theory [R. H. Kraichnan, “Turbulent thermal convection at arbitrary Prandtl number,” Phys. Fluids 5, 1374 (1962)] and the Grossmann–Lohse (GL) theory [S. Grossmann and D. Lohse, “Scaling in thermal convection: A unifying view,” J. Fluid Mech. 407, 27 (2000); “Thermal convection for large Prandtl number,” Phys. Rev. Lett. 86, 3316 (2001); “Prandtl and Rayleigh number dependence of the Reynolds number in turbulent thermal convection,” Phys. Rev. E 66, 016305 (2002); “Fluctuations in turbulent Rayleigh–Bénard convection: The role of plumes,” Phys. Fluids 16, 4462 (2004)], which both predict $\nu \sim R_a^{1/2}$ and $R_e \sim R_a^{1/2}$. However the Pr dependence within these two theories is different. Here we show that the numerical data are consistent with the GL theory $\nu \sim R_a^{1/2}$, $R_e \sim R_a^{1/2}$. For the thermal and kinetic dissipation rates we find $\epsilon_{\theta}/(\nu \Delta L^2) \sim (R_e R_a)^{0.77}$ and $\epsilon_e/(\nu^3 L^{-4}) \sim R_a^{2.77}$, both near (but not fully consistent) the bulk dominated behavior, whereas the temperature fluctuations do not depend on Ra and Pr. Finally, the dynamics of the heat transport is studied and put into the context of a recent theoretical finding by Doering et al. [“Comment on ultimate state of thermal convection” (private communication)]. © 2005 American Institute of Physics. [DOI: 10.1063/1.1884165]

I. INTRODUCTION

The scaling of large Rayleigh number (Ra) Rayleigh–Bénard (RB) convection has attracted tremendous attention in the last two decades. There is increasing agreement that, in general, there are no clean scaling laws for Nu(Ra,Pr) and Re(Ra,Pr), apart from asymptotic cases. One of these asymptotic cases has been doped the “ultimate state of thermal convection,” where the heat flux becomes independent of the kinematic viscosity $\nu$ and the thermal diffusivity $\kappa$. The physics of this regime is that the thermal and kinetic boundary layers have broken down or do not play a role any more for the heat flux and the flow is bulk dominated. The original scaling laws suggested for this regime are $\nu \sim R_a^{1/2}$, $R_e \sim R_a^{1/2}$. (1)

Relevant scaling laws are $\nu \sim R_a^{1/2}(\ln R_a)^{-3/2} R_a^{1/2}$. (2) for $Pr<0.15$, while for $0.15<Pr<1$, $\nu \sim R_a^{1/2} R_a^{-3/2} Pr^{-1/4}$. (3) $Re \sim R_a^{1/2}(\ln R_a)^{-3/2} Pr^{-1/4}$. (4)

The Grossmann–Lohse (GL) theory also gives such an asymptotic regime which is bulk dominated and where the plumes do not play a role (regimes IV and IV of Refs. 1–4). Apart from logarithmic corrections, it has the same Ra dependence as in Eqs. (1)–(4), but different Pr dependence, namely,

$\nu \sim R_a^{1/2} Pr^{1/2}$. (5)
\[ \text{Re} \sim Ra^{1/2}Pr^{-1/2}. \]  

The same scaling relation of Eq. (5) first appeared in the paper by Spiegel on thermal convection in stars,\(^{52}\) where it was proposed on the basis of the hypothesis that in high-turbulent conditions the dimensional heat flux shall be independent both of kinematic viscosity and thermal diffusivity. As a model of the ultimate regime we had suggested\(^{53}\) homogeneous RB turbulence, i.e., RB turbulence with periodic thermal gradient in the bulk respect to the imposed thermal gradient vs \(Ra\), the same scaling relation of Eq. (4) was not compatible with the asymptotic analysis. Let us denote the standard Boussinesq equation,

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + \nu \nabla^2 u + \beta g \frac{\nabla}{L} \theta. \]  

Here, \(\beta\) is the thermal expansion coefficient, \(g\) the gravity, \(p\) the pressure, and \(\theta(x,t)\) and \(u_i(x,t)\) are temperature and velocity field, respectively. This model has been previously studied by Borue and Orszag\(^{55}\) by means of a spectral numerical simulation with built-in hyperviscosity. They focused mainly on turbulent spectra and second-order correlation functions behavior, but not on scaling of integral quantities with respect to \(Ra\) and \(Pr\). Actually, their results were consistent with the suggested\(^{51,1–4}\) Ra dependence of Nu and Re, Nu \(\sim Ra^{1/2}\) and Re \(\sim Ra^{1/2}\). However, the Pr dependencies of Nu and Re, for which the predictions of Kraichnan\(^{51}\) and GL (Refs. 1–4) are different, has not yet been tested for homogeneous turbulence: this is the first aim of this paper (Sec. III). Section II contains details of the numerics. In Sec. IV we study the bulk scaling laws for the thermal and kinetic dissipation rates and compare them with the GL theory. In that section we study the temperature fluctuations \(\theta' = \langle \theta^2 \rangle^{1/2}\). The dynamics of the flow, including Nu(t) and its PDF (probability density function), is studied in Sec. V and put into the context of a recent analytical finding by Doering and co-workers.\(^{56}\) Section VI contains our conclusions.

II. DETAILS OF THE NUMERICS

Our numerical simulation is based on a lattice Boltzmann equation algorithm on a cubic 240\(^3\) grid. The same scheme and resolution has already been used in Refs. 54 and 57. We run two sets of simulations in statistically stationary conditions. The first fixed at \(Pr = 1\) varying the Ra number between \(9.6 \times 10^6\) and \(1.4 \times 10^7\). The second fixed at \(Ra = 1.4 \times 10^7\). This, the highest value we can reach at the present resolution, was studied for five different Pr numbers, 1/10, 1/3, 1, 3, and 4. We recorded shortly spaced time series of Nu and root mean squared (rms) values of temperature and velocity, and we stored a collection of the whole field configurations with a coarse time spacing. The length of each different run ranges between 64 and 166 eddy turnover times. Our simulation was performed on an APEmille machine in a 128 processor configuration.\(^{58,59}\) Each eddy turnover time requires on average 4 h of computation. The total computational time required for the whole set of simulations is roughly 150 days. The total number of stored configurations is around 2000.

III. Nu(Ra,Pr) AND Re(Ra,Pr)

The Nusselt number is defined as the dimensionless heat flux

\[ \text{Nu} = \frac{1}{\kappa A_{\text{eb}} L^{-1}} \langle (\delta_{\text{eb}} T)_{\text{A}}(z) - \kappa (\delta_{\text{eb}} T)_{\text{A}}(z) \rangle = \frac{\langle (\delta_{\text{eb}} T)_{\text{A}}(z) \rangle}{\kappa A_{\text{eb}} L^{-1}} - 1, \]  

where the average \(\langle \cdot \rangle_{\text{A}}\) is over a horizontal plane and over time. From Eqs. (7)–(10) one can derive two exact relations for the volume averaged thermal dissipation rate \(\epsilon_\theta = \kappa \langle (\delta_{\text{eb}} T)^2 \rangle_V\) and the volume averaged kinetic dissipation rate \(\epsilon_u = \nu \langle (\delta_{\text{eb}} u)^2 \rangle_V\), namely,

\[ \epsilon_u = \frac{L^2 \text{Nu Ra} Pr^{-2}}, \]
One can therefore numerically compute Nu in three different ways: (i) from its direct definition (10), (ii) from the volume averaged kinetic dissipation rate (11), and (iii) from the volume averaged thermal dissipation rate (12).

The results are shown in Fig. 1(a) as a function of Ra for \( \text{Pr}=1 \). There is very good agreement of Nu obtained from the three different methods for all Ra, giving us further confidence in the convergence of the numerics. If we fit all data points beyond Ra=10^5 with an effective power law, we obtain Nu \( \sim \text{Ra}^{0.50 \pm 0.05} \), consistent with the asymptotically expected law Nu \( \sim \text{Ra}^{1/2} \).\(^{61}\)

In Fig. 1(b) we display Nu as function of Pr for fixed Ra=1.4 \times 10^7. For the cases with Pr \( \neq 1 \) the convergence of the three different methods to calculate Nu is not perfect. This may be due to numerical errors in the resolution of the small scale differences, especially when \( \nu \) and \( \kappa \) are considerably different. However, one can clearly notice a strong increase of Nu with Pr. A fit with an effective power law gives Nu \( \sim \text{Pr}^{0.43 \pm 0.07} \), which is consistent with the asymptotic power law Nu \( \sim \text{Pr}^{1/2} \) suggested by the GL theory and by the small Pr regime \( (1) \) proposed by Kraichnan, but not with Kraichnan’s large Pr regime \( (3) \). Increasing Pr further (at fixed Ra) the flow will eventually laminarize, i.e., can no longer be considered as model system for the bulk of turbulence. This also follows from Fig. 2(b), in which we show the Reynolds number

\[
\text{Re} = \frac{u' L}{\nu}
\]

as function of Pr for fixed Ra=1.4 \times 10^7. Note that this is the fluctuation Reynolds number, defined by the rms velocity fluctuation \( u'=(u'^2)^{1/2} \), in homogeneous RB no large scale wind exists. \( \text{Re(Pr)} \) displays an effective scaling law \( \text{Re} \sim \text{Pr}^{-0.55 \pm 0.01} \), consistent with the GL prediction \( \text{Pr}^{-1/2} \) for the ultimate regime (if one identifies the wind Reynolds number in GL with the fluctuation Reynolds number here) and also with the Kraichnan prediction \( (2) \). Also the Ra scaling of Re is consistent with GL (and also with Kraichnan), \( \text{Re} \sim \text{Ra}^{1/2} \), as seen from Fig. 2(a) and as already shown in Ref. 53.

IV. SCALING LAWS FOR \( \epsilon_{\omega}, \epsilon_\theta \) AND THE TEMPERATURE FLUCTUATIONS

A. Kinetic and thermal dissipations

The homogeneous RB turbulence offers the opportunity to numerically test one of the basic assumptions of the GL theory, namely, that the energy dissipation rate in the bulk scales like

\[
\epsilon_\theta = \frac{\Delta^2}{L^2} \text{Nu}.
\]

FIG. 1. (a) Nu(Ra) for Pr=1, computed in three different ways: (●) using Eq. (10), (○) using Eq. (11), and (◻) from Eq. (12). The power law fits, performed on the mean value of the three different estimates and for Ra \( > 10^5 \), give a slope 0.50 \pm 0.05. (b) Nu(Pr) for Ra=1.4 \times 10^7, fit performed as before, with a resulting slope of 0.43 \pm 0.07.

FIG. 2. (a) Re(Ra) for Pr=1, with a fitted slope 0.50 \pm 0.02. (b) Re(Pr) for Ra=1.4 \times 10^7, with a fitted slope of −0.55 \pm 0.01.

In contrast if energy dissipation is dominated by the boundary layer (BL) region GL predicts the $\epsilon_{u,\text{BL}} \sim 4L^{-4}Re^{3/2}$ behavior. In Fig. 3(a) we plot $\epsilon_{u}/(v^3 L^{-4})$ vs $Re$ for all $Ra$ and $Pr$, and find $\epsilon_{u}/(v^3 L^{-4}) \sim Re^{2.77\pm 0.03}$ closer to the expectation (14) but not fully compatible with it.

The disentanglement of the thermal dissipation rate $\epsilon_\theta$ into two different scaling contributions is, in principle, less straightforward. The GL theory decombines

$$\epsilon_\theta/(k\Delta L^{-2}) = c_3(Re Pr)^{1/2} + c_4(Re Pr),$$

where the first term has been interpreted as boundary layer and plume contribution $\epsilon_{\theta,\text{pl}}$ and the second one as background contribution $\epsilon_{\theta,\text{bg}}$. The prefactors $c_3$ and $c_4$ are given in Ref. 4. Plumes are interpreted as detached boundary layer. Our simulation gives, see Fig. 3(b), the $\epsilon_\theta/(k\Delta L^{-2}) \sim (Re Pr)^{0.87\pm 0.04}$ behavior again closer to the background scaling, but not fully compatible with it, and consistent with the kinetic energy dissipation result. These unexpected deviations from the bulk behavior can be due to the presence of layers characterized by strong gradients both in the velocity and in the temperature field, i.e., to the formation of dynamical BL in the flow. It has been already observed in Ref. 55 and it is confirmed by our visualizations that the temperature field patterns often lead to the appearance of some nearby large vertical jets (see Sec. V of this paper). These jets are associated to the formation of strong vertical temperature gradients $(\partial_z \theta)$ on the surfaces at their boundaries. We think this feature of homogeneous convection is of basic importance to explain the observed deviations from pure bulk scaling.

### B. Temperature fluctuations

In our numerics we find the temperature fluctuations $\theta' = \langle (\theta')^2 \rangle^{1/2}$ to be independent from $Ra$ and $Pr$, see Fig. 4. These figures show that we have $\theta'/\Delta$ for all $Ra$ and $Pr$ within our numerical precision. In contrast, Ref. 4 predicted a dependence of the thermal fluctuations on both $Ra$ and $Pr$, namely, $\theta'/\Delta \sim (Pr Ra)^{-1/8}$ for the regimes IV$_L$ and IV$_S$ which correspond to the bulk of turbulence analyzed here. Our interpretation of Fig. 4 is that the bulk turbulence only has one temperature scale, namely, $\Delta$. For real RB turbulence it is the boundary layer dynamics which introduces further temperature scales, leading to the $Ra$ and $Pr$ number dependence of the temperature fluctuations observed in experiments.$^5,16,47,48$

### V. DYNAMICS OF THE FLOW

In this section we provide an insight into the dynamics of the periodic Rayleigh–Bénard flow. A bidimensional vertical snapshot of the flow is shown in Fig. 5. Already from this pictorial view the presence of an upward moving hot column and a downward moving cold column is clearly evident.

Indeed these large scale structure can be related to the presence of “elevator modes” (or jets, forming in the flow) growing in time until finally breaking down due to some instability mechanisms.

As proposed by Doering and co-workers in Ref. 56 it is possible to predict the presence of these modes directly start-
appearing for Ra, i.e., instability of the smallest possible wavenumber in the system arising from Eqs. (7) and (8). Doering et al. showed that, due to the periodic boundary conditions, this coupled system of equations admits a particular solution $\theta = \theta_0 e^{\lambda t} \sin(k \cdot x)$, $u_3 = u_0 e^{\lambda t} \sin(k \cdot x)$, $u_2 = u_0 = 0$, which is independent from the vertical coordinate $z$ [here $k = (k_x, k_y)$] and with

$$\lambda = -\frac{1}{2(Pr + 1)} k^2 + \frac{1}{2} \sqrt{(Pr + 1)^2 k^4 + 4 Pr \left( \frac{Ra}{L^2} - k^4 \right)}.$$  

(16)

From Eq. (16) one finds that the first unstable mode appears for $Ra = Ra_c = (2\pi)^4 \cdot 1558.54$, corresponding to the instability of the smallest possible wavenumber in the system, i.e., $k^2 = (2\pi/L)^2 n^2$ with $n = (1, 0)$.

The presence of accelerating modes with growth rate controlled by $\lambda$ can also be seen from Fig. 6 where we show Nu(t) on log scale (notice the huge range over which Nu fluctuates), and its logarithmic derivative.

In Fig. 7 we show the PDF of Nu(t) which is strongly skewed towards large Nu values. This asymmetry reflects the periods of exponential growth (also visible in Fig. 6). As can be seen in Fig. 8, for all Ra and Pr the system typically spends 54% of the time in growing modes.

Also the relative fluctuations of Nu on the Ra and Pr numbers (see Fig. 9) seem to indicate no dependencies, at least in the range of parameters studied.

Despite the presence of exact exploding solutions, our system clearly shows that in the turbulent regime these solutions become unstable due to some yet to be explored instability mechanism. The interplay between exploding modes and destabilization sets the value of the Nusselt number, i.e., the heat transfer through the cell.

VI. CONCLUSIONS

In conclusion, we confirmed that both the Ra and the Pr scaling of Nu and Re in homogeneous Rayleigh–Bénard convection are consistent with the suggested scaling laws of the Grossmann–Lohse theory for the bulk-dominated regime (regime IV of Refs. 1–3), which is the so-called ultimate regime of thermal convection. We also showed that the thermal and kinetic dissipations scale roughly as assumed in that theory. The temperature fluctuations do not show any Ra or Pr dependence for homogeneous Rayleigh–Bénard convection. From the dynamics the heat transport and flow visualizations we identify elevator modes which are brought into the context of a recent analytical finding by Doering et al. In future work we plan to further clarify the flow organization and, in particular, the instability mechanisms of the elevator modes which set the Nusselt number in homogeneous RB flow and therefore presumably also in the ultimate regime of thermal convection.
ACKNOWLEDGMENTS

The authors thank Charlie Doering for stimulating discussions on Sec. V. D. L. wishes to thank Siegfried Grossmann for extensive discussions and exchange over the years. This work was part of the research program of the Stichting voor Fundamenteel Onderzoek der Materie (FOM), which is financially supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO). Support by the European Union under Contract No. HPRN-CT-2000-00162 “Non Ideal Turbulence” is also acknowledged. This research was also supported by the INFN, through access to the APEmille computer resources. E.C. was supported by Neuricam spa (Trento, Italy) in the framework of a doctoral grant program with the University of Ferrara and during his visit at University of Twente by SARA through the HPC-Europe program.

In our previous paper 

\[ 10 \times 10 \text{ cm}^2 \]

we investigated the scaling behavior of heat transfer in turbulent Rayleigh–Bénard convection. Our main finding was that the heat flux \( q \) scales with the Rayleigh number \( R \) as

\[ q \propto R^{1+\varepsilon} \]

where \( \varepsilon \) is a scaling exponent that depends on the Prandtl number \( Pr \) and the Reynolds number \( Re \). This result extends the classical scaling law of Swift and Onslow (1962) by accounting for the effects of spatial confinement.

Our analysis relied on a detailed examination of the velocity fields and temperature fields measured in our experiments. We found that the heat transfer is dominated by fluid elements that are confined to corner regions of the enclosure, where the temperature gradient is steep. These elements contribute to the heat transport in a way that is different from the behavior observed in unconfined convection.

The scaling exponent \( \varepsilon \) was found to be

\[ \varepsilon = \left( \frac{1}{Pr} + \frac{1}{Re} \right) - 1 \]

which agrees well with the theoretical predictions of Verzicco and Camussi (1999) for Rayleigh–Bénard convection in a confined geometry.

Our results have implications for the design of thermal systems, where the efficient transfer of heat is crucial. The scaling law we derived can be used to predict heat transfer in other confined geometries, such as those found in microelectronics and miniature heat exchangers.

In conclusion, our study provides a deeper understanding of the heat transfer in confined convection, which is a fundamental process in many natural and industrial systems.